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Exam I , MTH 221, Summer 2018

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Score = $\frac{34}{34}$

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QUESTION 1. (4 points) Let $A = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$. Convince me that A is not diagnolizable.

1 linearly factored
$\alpha = 2, 2$

$C_A(\alpha) = (\alpha - 2)^2$, 2 E should be span of 2ind. p

$E_2 \Rightarrow [\alpha I_n - A]a = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 0 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 \in R$, $x_2 = 0$ Solution set $= \{(x_1, 0) \mid x_1 \in R\}$

Solution set $= \{x_1(1,0)\} \rightarrow$ span $\{(1,0)\}$ not diagnolizable becau only one ind. p but not 2

QUESTION 2. (4 points) Let A be a 4×4 matrix such that $C_A(\alpha) = (\alpha-2)^2(\alpha-3)^2$. Given $E_2 = span\{(3,0,0,0),(0,0,4,0)\}$ and $E_3 = span\{(0,2,0,0),(0,0,0,1)\}$. Find an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$. Then find the matrix A. [Hint: Stare really well and choose your D wisely!, then you may minimize the calculations!]

$\alpha = 2$, has Mult. 2 and $\alpha = 3$ has Mult. 2

$D = \begin{bmatrix} 2&0&0&0\\0&3&0&0\\0&0&2&0\\0&0&0&3 \end{bmatrix} \rightarrow Q = \begin{bmatrix} 3&0&0&0\\0&2&0&0\\0&0&4&0\\0&0&0&1 \end{bmatrix}$

$Q^{-1} = \begin{bmatrix} \frac{1}{3}&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{4}&0\\0&0&0&1 \end{bmatrix}$ $A = QDQ^{-1} = \begin{bmatrix} 3&0&0&0\\0&2&0&0\\0&0&4&0\\0&0&0&1 \end{bmatrix}\begin{bmatrix} 2&0&0&0\\0&3&0&0\\0&0&2&0\\0&0&0&3 \end{bmatrix}\begin{bmatrix} \frac{1}{3}&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{4}&0\\0&0&0&1 \end{bmatrix}$

in this case $A = \underbrace{QQ^{-1}}_{=I}D = QDQ^{-1}$

$A = \begin{bmatrix} 6&0&0&0\\0&6&0&0\\0&0&8&0\\0&0&0&3 \end{bmatrix}\begin{bmatrix} \frac{1}{3}&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{4}&0\\0&0&0&1 \end{bmatrix} = \begin{bmatrix} 2&0&0&0\\0&3&0&0\\0&0&2&0\\0&0&0&3 \end{bmatrix} = A = D$

QUESTION 3. (4 points) Given $A^{-1} = \begin{bmatrix} 1&1&1\\-1&0&0\\-2&-2&0 \end{bmatrix}$. Find the solution set to the system $A^T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\4\end{bmatrix}$

$(A^{-1})^T A^T \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = (A^{-1})^T\begin{bmatrix}0\\0\\4\end{bmatrix} \rightarrow \underbrace{(\cancel{A^{-1}A})^T}_{I_n}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = (A^{-1})^T\begin{bmatrix}0\\0\\4\end{bmatrix}$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = (A^{-1})^T\begin{bmatrix}0\\0\\4\end{bmatrix} \longrightarrow (A^{-1})^T = \begin{bmatrix} 1&-1&-2\\1&0&-2\\1&0&0 \end{bmatrix}$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix} 1&-1&-2\\1&0&-2\\1&0&0 \end{bmatrix}\begin{bmatrix}0\\0\\4\end{bmatrix} = \cancel{0\begin{bmatrix}1\\1\\1\end{bmatrix}} + \cancel{0\begin{bmatrix}-1\\0\\0\end{bmatrix}} + 4\begin{bmatrix}-2\\-2\\0\end{bmatrix} = \begin{bmatrix}-8\\-8\\0\end{bmatrix}$

Solution $= \{(-8, -8, 0)\}$ unique solution.

QUESTION 4. Let $A = \begin{bmatrix} 2 & 2 & 6 \\ -4 & -2 & -10 \\ -4 & -2 & -11 \end{bmatrix}$.

(a) (4 points) Find the LU-factorization of A.

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 6 & 1 & 0 & 0 \\ -4 & -2 & -10 & 0 & 1 & 0 \\ -4 & -2 & -11 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} 2R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 2 & 6 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 2 & 0 & 1 \end{array}\right] -R_2+R_3 \to R_3$$

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 6 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right] \to U = \begin{bmatrix} 2 & 2 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$\hookrightarrow L^{-1}$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 2 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}}_{U}$$

to find l start with I_3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} = L$$

(b)(4 points) Use (a) to find the solution set to $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$A\,Q = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow LUQ = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \to \overset{I}{\cancel{L^{-1}L}}U \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = L^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$U \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = L^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \cancel{0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$U \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \to \left[\begin{array}{ccc|c} 2 & 2 & 6 & 1 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$-1x_3 = -2 \to \boxed{x_3 = 1}$

$\to 2x_2 + 2x_3 = 3 \to 2x_2 + 2 = 3$

$2x_2 = 1 \to \boxed{x_2 = +\frac{1}{2}}$

$2x_1 + 2x_2 + 6x_3 = 1$

$2x_1 + 1 + 6 = 1$

$2x_1 = -6$

$x_1 = -3$

Solution set $= \{(-3, \frac{1}{2}, 1)\}$

QUESTION 5. Let $A = \begin{bmatrix} 1 & a & b \\ -1 & -a & 8 \\ -2 & 6 & c \end{bmatrix}$.

(a) (4 points) For what values of a, b, c will the matrix A be invertible?

invertible iff $|A| \neq 0$ $\alpha \neq 0$

$$\begin{bmatrix} 1 & a & b \\ -1 & -a & 8 \\ -2 & 6 & c \end{bmatrix} \begin{array}{l} R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 1 & a & b \\ 0 & 0 & b+8 \\ 0 & 2a-6 & 2b+c \end{bmatrix}$$

$a \neq -3$ $b \neq -8$

$c \in \mathbb{R}$

details on the back side of previous page

(b) (2 points) For what values of a, b, c will the system $A^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ have unique solution?

$|A| = |A^T| \neq 0$

$a \neq -3$ $b \neq -8$ $c \in \mathbb{R}$

a)

$|A| = 0$

$$\begin{bmatrix} 1 & a & b \\ -1 & -a & 8 \\ -2 & 6 & c \end{bmatrix}$$

$$\xrightarrow[2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 1 & a & b \\ 0 & 0 & b+8 \\ 0 & 2a+6 & 2b+c \end{bmatrix} \rightarrow$$

row equivalent to A which the transpose of it is row equivalent of A^T

$$(b+8)\begin{vmatrix} 1 & a \\ 0 & 2a+6 \end{vmatrix} = (b+8)(2a+6)$$

$b \neq -8 \quad a \neq -3 \quad c \in \mathbb{R}$

b)

$$A^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$|A^T| \neq 0$

$A^T \underset{\text{row equivalent}}{\sim} I_n$

$|A^T| = |A|$

$$A^T = \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ a & -a & 6 & 0 \\ b & 8 & c & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 2a-6 & 0 \\ b & b+8 & 2b+c & 0 \end{array}\right]$$

$|A^T| = -(2a+6)(b+8)$

$a \neq -3 \quad b \neq -8$

$c \in \mathbb{R}$

QUESTION 6. (8 points) Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 5 \\ 0 & 1 & -4 \end{bmatrix}$. If A is diagnolizable, then find an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$.

1) $C_A(\alpha)$ linearly factored

$|\alpha I_n - A| = 0 \rightarrow \alpha I_n - A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 5 \\ 0 & 1 & -4 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ -1 & \alpha & -5 \\ 0 & -1 & \alpha+4 \end{bmatrix}$

$|\alpha I_n - A| = \alpha\left[\alpha(\alpha+4) - 5\right] = \alpha\left[\alpha^2 + 4\alpha - 5\right]$

$= \alpha^3 + 4\alpha^2 - 5\alpha = (\alpha-1)(\alpha+5)(\alpha)$

$\alpha = 1, -5, 0$

linearly factored ✓

2) each, has span of 1 ind. point

Yes, because A is nxn and there is n distinct eigen values so it is diagnolizable.

plug in here

$[\alpha I_n - A] Q^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$E_1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -5 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & -5 & 0 \\ 0 & -1 & 5 & 0 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & -1 & 5 & 0 \end{array}\right]$

$\xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 - 5x_3 = 0$, $\therefore x_2 = 5x_3$, $x_3 \in \mathbb{R}$

$E_1 = \text{Span}\{(0, 5, 1)\}$

$-5 \rightarrow \left[\begin{array}{ccc|c} -5 & 0 & 0 & 0 \\ -1 & -5 & -5 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \overset{-\frac{1}{5}R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & -5 & -5 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] -\frac{1}{5}R_2$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \overset{R_2+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Read $x_1 = 0$, $x_2 + x_3 = 0$, $\therefore x_2 = -x_3$, $x_3 \in \mathbb{R}$

$E_{-5} = \text{Span}\{(0, -1, 1)\}$

$0 \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & -5 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right]$ $-x_1 - 5x_3 = 0$, $\therefore -x_1 = 5x_3$, $x_1 = -5x_3$; $-x_2 + 4x_3 = 0$, $x_2 = 4x_3$; $x_3 \in \mathbb{R}$

$E_0 = \text{span}\{(-5, 4, 1)\}$

for Q & D see on the back side of beside page

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lind. p each ∴ diagnolizable

$E_1 = \text{span}\{(0,5,1)\}$ $\quad \alpha = 1$

$E_{-5} = \text{span}\{(0,-1,1)\}$ $\quad \alpha = -5$

$E_0 = \text{span}\{(-5,4,1)\}$ $\quad \alpha = 0$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore \quad Q = \begin{bmatrix} 0 & 0 & -5 \\ 5 & -1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A = Q D Q^{-1}$$

A is Not invertible because $\alpha = 0$